Local and Global Well-posedness for the Critical Schrödinger-Debye System

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The Schrödinger-Debye System

The Cauchy problem for the Schrödinger-Debye system is:

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = uv, \\ \mu\partial_t v + v = \lambda |u|^2, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x). \end{cases}$$

where, for $n = 1, 2, 3$

$$u : \mathbb{R}^n_x \times \mathbb{R}_t \to \mathbb{C}, \qquad v : \mathbb{R}^n_x \times \mathbb{R}_t \to \mathbb{R},$$

and

$$\mu > 0 \qquad \lambda = \pm 1.$$

The Schrödinger-Debye system models the propagation of an electomagnetic wave in a non-resonant medium where the response time is relevant.

The second equation, for the real function $v(x,t)$

$$\mu \partial_t v + v = \lambda |u|^2,$$

is just an ODE which can be easily solved

$$v(x,t) = e^{-t/\mu} v_0(x) + \frac{\lambda}{\mu} \int_0^t e^{-(t-t')/\mu} |u(x,t')|^2 \, dt'$$

decoupling the original system into just an integro-differential equation

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = e^{-t/\mu} v_0 u + \frac{\lambda}{\mu} u \int_0^t e^{-(t-t')/\mu} |u(t')|^2 \, dt', \\ u(x,0) = u_0(x). \end{cases}$$

The Cubic Nonlinear Schrödinger equation (cNLS)

In the case $\mu = 0$ (absence of delay) the system reduces to the celebrated cubic NLS equation

$$i\partial_t u + \frac{1}{2}\Delta u = \pm|u|^2 u$$

where

$$u : \mathbb{R}^n_x \times \mathbb{R}_t \to \mathbb{C},$$

and the equation is classified, depending on the sign of the nonlinearity, as

$$\begin{cases} \text{Focusing:} & \lambda = -1 \\ \text{Defocusing:} & \lambda = +1 \end{cases}$$

Well posedness results for the cNLS equation

Recall that the scaling invariance for the cNLS is given by

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$$

and therefore the scaling criticial Sobolev index is

$$\boxed{s_c = \frac{n}{2} - 1}$$

- Local Well-posedness
  - J. Ginibre, G. Velo [J. Funct. Anal., 1979] proved LWP for the (subcritical) cases $s = 1$ and $n = 1, 2, 3$.
  - Y. Tsutsumi [Funk. Ekva., 1987] proved LWP for the (subcritical) case $s = 0$ and $n = 1$.

- Local Well-posedness
  - T. Cazenave, F. Weissler [Lecture Notes in Math, 1989] proved LWP for the critical case $s = 0$ and $n = 2$.
  - T. Cazenave, F. Weissler [Nonlinear Anal. T.M.A., 1990] proved LWP for the fractional critical, and subcritical, exponents $s \geq \max\{0, s_c\}$, for $n \geq 1$.
- Global Well-posedness
  - In the L^2 subcritical case $(n = 1)$ global existence is an immediate consequence of the LWP result and the L^2 conservation.
  - In the defocusing $(\lambda = +1)$ and H^1 subcritical cases $(n = 1, 2, 3)$ global existence is an immediate consequence of the LWP result and the energy conservation.
  - In the critical cases L^2 $(n = 2)$ and $H^{1/2}$ $(n = 3)$ there is global existence for small initial data.

Blow-up results for the cNLS equation in H^1

In the focusing $(\lambda = -1)$ case

$$i\partial_t u + \frac{1}{2}\Delta u = -|u|^2 u$$

- The Gagliardo-Nirenberg inequality and the energy conservation guarantee global existence for arbitrary H^1 initial data only as long as the problem is L^2 subcritical (n=1).
- For the L^2 critical case (n=2) the optimal constant in the Gagliardo-Nirenberg inequality shows that if the L^2 norm of the H^1 initial data is sufficiently small - smaller than the mass of the standing wave - the H^1 solution is global.
- For the L^2 supercritical, H^1 subcritical case (n=3) the solution is global only if the H^1 initial data is small.

For the defocusing cNLS there is actually blow-up of the H^1 solutions in the L^2 critical and supercritical regimes $(n = 2, 3)$.

In these regimes the virial inequality is

$$\frac{d^2}{dt^2} \int |x|^2 |u(x,t)|^2 dx \leq 8nE_0$$

where E_0 is the conserved energy of the cNLS

$$E_0 = \frac{1}{2} \int |\nabla u|^2 dx - \frac{1}{4} \int |u|^4 dx.$$

Thus, by making $\|u_0\|_{H^1}$ large enough, one can always achieve $E_0 < 0$ which implies finite time blow up of the H^1 solution.

Conserved Quantities of the Schrödinger-Debye System

The solutions of the Schrödinger-Debye system satisfy conservation of the L^2 norm

$$\int |u(x,t)|^2 dx = \int |u_0(x)|^2 dx$$

and the pseudo-Hamiltonian structure

$$\frac{d}{dt}E(t) = 2\lambda\mu \int_{\mathbb{R}^N} (v_t)^2 dx,$$

where

$$E(t) = \int_{\mathbb{R}^N} \{|\nabla u|^2 + \lambda |u|^4 - \lambda\mu^2 (v_t)^2\} dx = \int_{\mathbb{R}^N} \{|\nabla u|^2 + 2v|u|^2 - \lambda v^2\} dx.$$

Well-posedness for the Schrödinger-Debye System

$n = 1, 2, 3$

In 1998 and 2000, B. Bidégaray established the following local existence results, using the Strichartz estimates for the unitary Schrödinger group applied to the decoupled integro-differential equation

Theorem: Let $n = 1, 2, 3$ and $(u_0, v_0) \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$. Then, for small enough $T = T(\|u_0\|_{H^s}, \|v_0\|_{H^s})$, the initial value problem for the Schrödinger-Debye system has a unique solution

(a) $u \in L^\infty([0,T]; H^s(\mathbb{R}^n))$ if $s > n/2$,

(b) $u \in L^\infty\big([0,T]; H^1(\mathbb{R}^n)\big)$ if $s = 1$,

(c) $u \in C\big([0,T]; L^2(\mathbb{R}^n)\big) \bigcap L^{8/n}\big([0,T]; L^4(\mathbb{R}^n)\big)$ if $s = 0$.

Finally, if $(u_0, v_0) \in H^2(\mathbb{R}^n) \times L^4(\mathbb{R}^n)$, there also exists a unique solution $(u, v) \in C\big([0,T]; H^2(\mathbb{R}^n) \times L^4(\mathbb{R}^n)\big)$.

$n = 1$

In 2009, A. Corcho and C. Matheus established the following, in the framework of Bourgain spaces

Theorem: For any $(u_0, v_0) \in H^s(\mathbb{R}) \times H^\ell(\mathbb{R})$, where

$$|s| - 1/2 \leq \ell < \min\{s + 1/2,\ 2s + 1/2\} \quad \text{and} \quad s > -1/4,$$

there exists a time $T = T(\|u_0\|_{H^s}, \|v_0\|_{H^\ell}) > 0$ and a unique solution $(u(t), v(t))$ of the initial value problem in the time interval $[0, T]$, satisfying

$$(u, v) \in C\left([0, T]; H^s(\mathbb{R}) \times H^\ell(\mathbb{R})\right).$$

In addition, in the case $\ell = s$ with $-3/14 < s \leq 0$, the local solutions can be extended to any time interval $[0, T]$.

$n = 2, 3$

Using Bourgain space techniques analogous to J. Ginibre, Y. Tsutsumi, G. Velo [J. Funct. Anal. 1997], for the Zakharov system, we established the following LWP result

Theorem: Let $n = 2, 3$. For any $(u_0, v_0) \in H^s(\mathbb{R}^n) \times H^\ell(\mathbb{R}^n)$, with s and ℓ satisfying the conditions:

$$\max\{0, s-1\} \leq \ell \leq \min\{2s, s+1\}$$

there exists a positive time $T = T(\mu, \|u_0\|_{H^s}, \|v_0\|_{H^\ell})$ and a unique solution $(u(t), v(t))$ of the initial value problem on the time interval $[0, T]$, such that

$$(u, v) \in C\left([0, T]; H^s(\mathbb{R}^n) \times H^\ell(\mathbb{R}^n)\right).$$

Obs: The cases $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n),\ s \geq 0$ and $H^{s+1}(\mathbb{R}^n) \times H^s(\mathbb{R}^n),\ s \geq 0$ are included in this theorem.

The idea of the proof consists, not in decoupling the system, but as usual in writing it in integral form through Duhamel's formula

$$\begin{cases} u(t) = S(t)u_0 - i \displaystyle\int_0^t S(t-t')uv(t')\,dt' \\ v(t) = e^{-t/\mu}v_0 + \frac{\lambda}{\mu} \displaystyle\int_0^t e^{-(t-t')/\mu}|u(t')|^2\,dt' \end{cases}$$

where $S(t) = e^{it\Delta/2}$ denotes the Schrödinger unitary propagator.

The solution to the system is obtained by applying a Picard iteration scheme to this integral formulation, showing that it contracts to a fixed point in appropriate Bourgain spaces with time exponents $> 1/2$.

$$\begin{cases} \|u\|_{X^{s,b}} = \| < \xi >^s < \tau + \frac{1}{2}|\xi|^2 >^b \hat{u}(\xi,\tau)\|_{L^2_{\xi,\tau}}, \\ \|v\|_{H^{l,c}} = \| < \xi >^l < \tau >^c \hat{v}(\xi,\tau)\|_{L^2_{\xi,\tau}}. \end{cases}$$

The proof typically reduces to proving multilinear estimates for the nonlinear terms.

For the previous LWP result this was obtained by establishing the following bilinear inequalities.

$$\|uv\|_{X^{s,-1/2+}} \lesssim \|u\|_{X^{s,1/2+}} \|v\|_{H^{l,1/2+}} \qquad s \geq 0, \quad l \geq \max\{0, s-1\}$$

and

$$\|u\bar{w}\|_{H^{l,-1/2+}} \lesssim \|u\|_{X^{s,1/2+}} \|w\|_{X^{s,1/2+}} \qquad s \geq 0, \quad l \leq \min\{2s, s+1\}$$

Global Well-posedness for the Critical Model

$n = 2$

Theorem: Let $(u_0, v_0) \in H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$. Then, for all $T > 0$, there exists a unique solution

$$(u, v) \in C\left([0, T];\ H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)\right)$$

to the initial value problem associated to the Schrödinger-Debye system (both, for the defocusing ($\lambda = 1$) as well as focusing ($\lambda = -1$) cases).

The idea of the GWP proof is based on obtaining an a priori bound for the quantity

$$f(t) := \|\nabla u(\cdot,t)\|_{L^2}^2 + \|v(\cdot,t)\|_{L^2}^2$$

which, together with the conservation of the L^2 norm of u, yields control of the full norm $\|u(\cdot,t)\|_{H^1} + \|v(\cdot,t)\|_{L^2}$.

Now, the term $\|v(\cdot,t)\|_{L^2}^2$ is controlled by the explicit formula

$$v(x,t) = e^{-t/\mu} v_0(x) + \frac{\lambda}{\mu} \int_0^t e^{-(t-t')/\mu} |u(x,t')|^2 \, dt'$$

whereas the term $\|\nabla u(\cdot,t)\|_{L^2}^2$ is controlled through

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx = E(t) - \int_{\mathbb{R}^N} \{2v|u|^2 - \lambda v^2\} dx,$$

The only particular ingredient is the use of the Gagliardo-Nirenberg inequality, for $n = 2$,

$$\|u\|_{L^4} \leq C_{GN} \|\nabla u\|_{L^2}^{1/2} \|u\|_{L^2}^{1/2}.$$

For example, using the explicit formula for $v(x, t)$

$$\begin{aligned}
\|v(\cdot, t)\|_{L^2} &\leq \|v_0\|_{L^2} + \tfrac{1}{\mu} \int_0^t e^{-(t-t')/\mu} \|u(\cdot, t')\|_{L^4}^2 \, dt' \\
&\leq \|v_0\|_{L^2} + \tfrac{C_{GN}^2}{\mu} \int_0^t \|u(\cdot, t')\|_{L^2} \|\nabla u(\cdot, t')\|_{L^2} \, dt' \\
&= \|v_0\|_{L^2} + \tfrac{C_{GN}^2 \|u_0\|_{L^2}}{\mu} \int_0^t \|\nabla u(\cdot, t')\|_{L^2} \, dt',
\end{aligned}$$

which, after squaring and using Hölder, becomes

$$\|v(\cdot,t)\|_{L^2}^2 \le 2\|v_0\|_{L^2}^2 + 2\left(\frac{C_{GN}^2\|u_0\|_{L^2}}{\mu}\int_0^t \|\nabla u(\cdot,t')\|_{L^2}\,dt'\right)^2$$

$$\le 2\|v_0\|_{L^2}^2 + \frac{2C_{GN}^4\|u_0\|_{L^2}^2}{\mu^2}\,t\int_0^t f(t')\,dt'.$$

Then, this estimate of $\|v(\cdot,t)\|_{L^2}$ is used in the pseudo-Hamiltonian structure equation for $E(t)$, to finally produce the full a priori bound for $f(t)$

$$\boxed{f(t) \le \alpha_0 + \alpha_1\int_0^t f(t')dt', \quad \text{for all} \quad t \in [0,\, T_\mu]}$$

where $\alpha_0 = \alpha_0(\|u_0\|_{L^2}, \|v_0\|_{L^2})$, $\alpha_1 = \alpha_1(\|u_0\|_{L^2})$ are constants and

$$\boxed{T_\mu = \frac{\mu}{4C_{GN}^4\|u_0\|_{L^2}^2}}$$

depends only on the conserved quantity $\|u_0\|_{L^2}$.

Observations:

- The solution can grow very rapidly and explode at t=∞.
- In $n = 2$ the H^1 norm barely fails to control L^∞ therefore this does not rule out blow up of $\|u\|_{L^\infty}$.
- In $n = 2$ and initial data in $H^1 \times H^1$ the previous global result shows that blow up can only occur for $\|\nabla v\|_{L^2}$.
- In $n = 1$ the previous proof can be easily modified to prove global existence in $H^1 \times H^1$, for which the work of Corcho & Matheus already provided LWP.

That's All Folks!